

Towards a General Theory of the Combination of Logics

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Abstract. The purpose of this article is to present some general concepts and problems related to the combination of logics.² A brief history of the combination of logics is presented with the aim of finding unity in the methods for combining logics. Some general notions are analyzed (for instance, methods entail methods) - as well as some problems: the paradox of the combination of logics and the collapsing problem. Despite the existence of different methods for combining logics, is it possible to come up with a universal and general approach able to unify the subject? I propose the *powerful method problem* and argue that a positive answer to this question can also be seen as an initial clue towards a general theory on the combination of logics.

1. Introduction. Questions about the nature of logical systems and logics are controversial and there is no global agreement related to the following problems: (A) What is a logical system? (B) What is a logic? Despite the plurality of answers, it is possible to *assume* that a logic is a pair composed by a set of propositions, indeed an algebra of propositions, and a consequence relation (i.e. logical consequence) on this set ³. Not only are there many definitions of logic but also there is not just only

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²I decided to write this article while I was reading the tutorial given by C. Caleiro in the *First World Congress and School on Universal Logic*. Caleiro says the following: "We adopt a methodological abstract viewpoint that is concerned with general universal mechanisms for combining logics. Rather than focusing on the specific details of the combination of particular logics, we aim at rigorously defining a logic combination mechanism at the adequate level of abstraction and then establishing meaningful transference results that may be used in many situations. The typical questions to be asked and answered are: (i) When does it make sense to combine two given logics and what is the result? (ii) If two logics with property P are combined does the resulting logic inherit the property P?"

³A consequence relation without restriction means that it does not need to respect the Tarskian axioms for logical consequence

one logic. The twentieth century has shown that logics are like some kind of computer virus, in the sense that each day a new one appears. To understand all these varieties of logics, it was urgent to find a way to unify this multiplicity. In this sense, *universal logic* was created in order to study general properties of logical systems (see [1]). Universal logic is a general theory of logical systems motivated basically by the proliferation of the logical systems available in the logical land. Universal logicians try to develop abstract tools which can be used to understand a plurality of logics from an abstract viewpoint instead of investigating a particular logic.⁴

In the same way that there are different logics, there are also many methods to combine logics: temporalization, synchronization, fusion, product, fibring etc. There is a vast literature about each one of these methods. They are useful in the sense that they allow us to better understand complex statements in natural languages as well as helping us understand problems related to technical fields like computer science (see [13]).

Indeed, combining logics is not an easy task, assuming that there are many definitions of the logical systems, as well as of different logics, and that there are many different presentations of the same logic: sequent calculus, natural deduction, tableaux, different semantics, etc. For this reason, it is urgent to develop a general theory on the combination of logics, able to answer, at least, one basic question: What is the definition of the combination of logics?

Such a theory must also be able to answer the following questions: How to unify all these methods for combining logics? What general properties are inherent in all of them? Is there a general theory about methods for combining logics? This article gives a clue about how to answer these questions applying a universal approach to the combination of logics.

2. A brief history of the combination of logics. There are many different perspectives on combination. One may want to combine two given logics, combining all their operators or one may want add a particular feature to a given logic, for example: temporalization - adding the concept of time to a given logic - fuzzification - adding a fuzzy character to a logic - paraconsistentization of logics - given a logic, how to obtain the paraconsistent counterpart of this logic. There are also different kinds of decomposition of a given logic into fragments such as for example possible-translation

⁴It seems to me that universal logic is a kind of platonism, given that it accepts that logics, despite of their different manifestations, have something in common, a kind of essence. It is the same idea but now applied to logics.

semantics [4] and [12].

At the beginning, combination of logics appeared in the environment of modal logic and, therefore, many methods were specially created to model on the one hand combination of Kripke Structures, on the other hand combination of axiomatic systems, although nowadays they are applied to a great variety of model-theoretical and proof-theoretical notions. The ideal would be to look for a universal conception of logical structure and thus to define an abstract method for combining logics independently of any particular conceptions of a logic.

The simplest method introduced to combine modal logics was fusion (see [17]). Semantically speaking, fusion consists in putting together without interferences two Kripke semantics, that is to say putting side by side the accessibility relations. Fusion of Hilbert proof systems also consists just of putting together rules and axioms. The basic idea underlying fusion is the combination of the languages of the two logics, done in a natural way, the rest follows straightforwardly. Fusion generally preserves soundness, completeness, finite model property etc.

Another method, much more complicated, is product. It was first used by Shehtman to introduce a two-dimensional modal logic. Product defines the notion of dimension in logics.⁵ Fusions and products have been very useful in modal logics⁶ and in issues related to the modelling of philosophical concepts. A difficult and interesting task is to determine the proof-theoretical counterpart of a product of models (see [15])⁷, i.e. which method for combining proof-theoretical systems preserves completeness for product of semantics.

Even fibring, the most famous method, was developed in the context of modal logics. According to Dov Gabbay, who proposed fibring, this mechanism allows us to associate to each possible world a model using a fibring function. Fibring is a method used to combine logics while evaluating a formula which has an operator that can not be recognized in a particular language. Fibring is considered the most powerful method of combining logics because it allows, in one of its variations, interactions between languages. Gabbay uses an idea of fibring which is based on the idea of a fibring function, and on complex models with a fibring function - however, nowadays there are different definitions of fibring which are strongly related to

⁵Not just modal logics, but logics in general.

⁶A nice presentation of multi-dimensional modal logics, and of how it is obtained by products, is presented in [18].

⁷For many cases it is known.

different conceptions of logic (See for example the works of the Portuguese school and the importance of selecting the right level of abstraction or the working universe of logics.) To determine if metalogical notions as soundness and completeness are preserved by fibring depends strongly on a given conception of logic and fibring (see [11]).

3. Combining logics. A logician who combines logics performs a task similar to that of chemists, but instead of atoms and molecules, the logicians deal with languages, models and logics. The chemist has tools to realize the process of combining substances and afterwards he/she separates them. And the logicians also have their own methods and techniques for combining logics and decomposing logics.

When logicians decide to combine logics they have to select the correct level of abstraction related to the nature of logics. It does not make sense to combine logics without stating, first, what is *assumed* as a logic.

To explain the problem let me mention the works developed by the Portuguese school, especially [11]. The simplest type of logic considered is called a *consequence system*, which is a pair composed of a set and a consequence operator obeying Tarski's axiom. However, this kind of consequence system does not constitute the right level of abstraction to combine logics, because the structure of the formulas is not defined, but it can be useful to introduce the combination of logics in a high level of abstraction. The next natural step is to determine the structure of the formulas and to work with a structural consequence operator in the sense of Łoś and Suszko. The Portuguese School go further by defining the notion of *deductive systems*, which is a structural consequence operator together with a set of inference rules [11]. Then they define the fibring of deductive systems. On the other hand they also introduce the concept of *interpretation systems* in order to combine semantics. An *interpretation system* is a structural consequence operator together with a set of interpretation structures, i.e. models [11]. They then define fibring of interpretation systems. To define what a logical system is, they make use of both notions. From [11] we learn that: logical systems are obtained by putting together deductive systems and interpretation systems in order to create a nice environment to talk about soundness and completeness. Therefore, logical systems are a good level of abstraction to begin to define particular combination methods between logics. Almost all results obtained in the combination of logics depend strongly on particular conceptions of the logical systems and of the methods for combining

logics.⁸

In the same way that a theory of truth should be able to answer the question about what truth is and what criteria are used to determine when a proposition is true, a general theory on the combination of logics should be able to answer the question about what combination is and what the procedure is for combining logics⁹

There are many different methods of combination: fusion, product, fibering, synchronization, etc, each one applying to some particular proof systems, semantical structures, logical structures incorporating or not proof or semantical features. However, up to now there is no general clear definition of what combination of logics is, independent of circumstances.

Gabbay in [16] says that the combination of two logics is the least conservative extension defined on the combined language, but as Caleiro noted that two logics may not have a common conservative extension. Beziau gave a simple example: the combination of a logic with only a classical negation and a logic with only an intuitionistic implication. In the combined logic, the intuitionistic implication becomes classical, therefore we don't have a conservative extension.

4. General questions about combination of logics. Instead of investigating particular conceptions of logics and particular methods for combining logics, we can inquire what are the general problems related to the combination of logics.

4.1. Preservation of properties. The first, and one of the most important and popular problems in the combination of logics, is the question about the *preservation of properties* or the *transfer theorems*. This question can be stated as follows: given two logics each having the property P , which methods of combination of these two logics preserve this property, i.e. produce a combined logic which is P ? This property can be truth-functionality, completeness, the finite model property, etc.¹⁰

To give an answer to the question above we should have information about the nature of the logics considered and about the nature of the method used in the combination. This problem - let us call it the *preserving*

⁸I mentioned the article [11] just as an example, but these techniques appear in all the works of the Portuguese school, see for example Caleiro's PhD thesis [3].

⁹Carnielli and Coniglio have tried to give a categorial definition of combining logics [5].

¹⁰Concerning the preservation of properties (soundness, completeness etc) related to particular methods I recommend [8, 9, 15, 16, 17].

properties problem - is a good example of a universal question related to the combination of logics. As it is known, the process of combining logics can be realized in two different directions. The first one is the combination, literally, of logics. The second one is the decomposition of logics. The same general problem above applies also in the last case. Given a complex logic, if it is decomposed, how goes the preservation of its properties for its fragments?

4.2. Categorical Representation. Instead of investigating particular objects and particular transformations between these objects, the pure categorist studies general constructions with categories. Logical systems can be viewed as objects of categories. In this sense it is possible to construct categories of deductive systems, of interpretation systems and of logic systems [11].

An interesting problem that arises in questions about the combination of logics is the categorical representation of a given method. For example, in [9] it is proved that fibring is a kind of universal construction in a particular category. The second general problem related to the combination of logics is the *categorical representation problem* and can be stated as follows: *Given a method for combining logics, does this method have a universal construction in a category?*

4.3. Methods entail Methods. The third problem related to our universal approach to the combination of logics is the *methods entail methods problem*. This problem is a clue in the direction of finding a powerful and universal method for combining logics. When one selects a particular method for combining logics, would there be another method that could be generated by this method? To illustrate this problem, note that in [8] the authors show a plurality of relations between fibring, synchronization and parametrization and show how we can deduce one from another. This is a clear example of the *methods entail methods problem*, which can be stated as: *Given a method for combining logics, does this method imply other methods?*

5. Paradoxes related to the combination of logics. Although the fact that combining logics allows us to get more powerful logics, there are some problems related to the basic concepts in the subject.

5.1. The collapsing paradox. Gabbay has pointed out (see [15]) that the fibration of two logics leads to collapse. For example the fibring of classical implication and intuitionistic implication logic leads to collapses

into classical implication. Logicians are developing many variations of fibring trying to solve this problem (see for instance [14] and [10]).

5.2. The copulation paradox. Beziau has pointed out a interesting problem arising with combination of logics [2]. It deals with combination of truth-tables. If one puts together in a natural way the standard semantics of conjunction and the standard semantics of disjunction, one gets a logic in which distributivity holds between conjunction and disjunction, so the combined logic produced by this combination of semantics is not the least conservative extension of the logic of conjunction and the logic of disjunction. Beziau calls this phenomenon by the suggestive name “copulation paradox”, because the conjunction and the disjunction are interacting and producing a new property.

6. Conclusion. A very important task in the combination of logics is to find the right level of abstraction for logical structures. Logicians usually prefer to speak about logical structures where it is possible to express syntactical and semantical properties of the logics like, for instance, soundness and completeness. After deciding about the best way to express what a logic is, it is then possible to define operations between these logics as, for example, fibring. In this sense, it is possible to define many varieties of fibring: fibring of deductive systems, interpretation systems, logical systems presentation and so on [11]. Methods for combining logics are tools which can be used by the universal logician to find again a unity in logic.

The *powerful method problem* is the intersection of the three general problems presented in this paper:

- 1) Is it possible to find a universal method able to show that most of all known methods are particular cases?
- 2) Is it possible to give a categorial characterization of this method?
- 3) What are the properties preserved by this method?

A positive answer to the above questions would probably be a paradise for those who are working in the combination of logics. Many people think that fibring would be a solution to this problem, but there are some problems with fibring such as the collapsing problem. This problem leads many logic combinatorators to propose new kinds of fibring as for example [10] and [14]. Despite these proposals, we do not have any guarantee that the powerful method problem is already solved.

In order to combine logics it should first be clear what a logic is. How-

ever, there is not a unique answer to this question. The most that one can do is suppose that a logic is something in particular and see what follows from the supposition. It is reasonable enough to depart from a conception of logic which permits us to speak about the two sides of a logical system: its syntax and semantics. Afterwards, we have to be able to determine exactly what are the properties of an abstract method for combining logics. Using a method for combining logics, we can enrich our languages and consequently our logics, being able thus to better understand formal languages and their applications.

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