Combining possibility and knowledge

Alexandre Costa-Leite

Institute of Logic - Neuchâtel - Switzerland

Abstract

This paper is an attempt to define a new modality with philosophical interest by combining the basic modal ingredients of possibility and knowledge. This combination is realized via product of modal frames so as to construct a *knowability* modality, which is a bidimensional constructor of arity one defined in a two-dimensional modal frame. A semantical interpretation for the operator is proposed, as well as an axiomatic system able to account for inferences related to this new modality. The resulting logic for knowability LK is shown to be sound and complete with respect to its class of modal-epistemic product models.

1 Introduction

According to [4], "as logic is being used more and more to formalise field problems in philosophy, language, artificial intelligence, logic programming and computer science, the kind of logics required become more and more complex."In this paper a particular method for combining logics is used to construct a logic able to express a philosophical concept as the concept of knowability, which has been a target of several articles related to the knowability paradox. This paradox is a modal argument which shows that given the monomodal logic KT extended with the verificationist principle $A \rightarrow \diamond KA$ it is possible to deduce $A \rightarrow KA$ causing the collapse of the knowledge modality. The motivation of this article is to propose a modal formalism able to account for the complex knowability modality which emerges in the presence of the verificationist principle. A natural conjecture underlying this proposal is that using the knowability modality it is possible to define an n-dimensional modal version of the logic KT where it is possible to add a bidimensional version of the verificationist principle without causing the collapse of the knowledge operator. This would be a nice solution to the trouble. Up to now, there is no definitive argument in this direction, but just a general clue.

The basic difference between the knowability modality and other usual modalities, in general terms, is that it is not possible to define knowability in monodimensional modal reality. Therefore, a knowability modality requires a two-dimensional environment defined by means of monomodal frames.

There are several different methods for combining modal logics: fibring [5] or synchronization as defined in [2] or, otherwise, one can use fusions [6] or products as defined in [4]. Also, given a particular method for combining logics, it is possible to give a categorial representation of the mechanism as in [3].

In this article a concrete case of product of modal frames is used to define a two-dimensional modal frame where it is possible to express a combined modality of *knowability*. In order to realize this task, the product of two modal frames (interpreted as alethic and epistemic, respectively) is defined as in [5]. Technical details of the construction will appear in the final version of the paper.

2 Bidimensional modal semantics for the knowability operator

It is well-known that the operation of product when applied to singular modal frames defines many-dimensional modal frames. Many-dimensional frames are used to interpret modal languages for multi-dimensional modal logics [8]. In [5] there is the definition bellow, which is adequate for our account of knowability.

Definition 1 *Consider two generalized frames:*

$$F_1 =< W, R_1, ..., R_n >$$
$$F_2 =< S, P_1, ..., P_n >$$

Then, a simple product of generalized frames *is defined as*

$$F_1 \times F_2 =< W \times S, R'_1, ..., R'_n, P'_1, ..., P'_n >$$

such that:

$$R'_i = \{<< x, y >, < z, y >>: xR_iz, y \in S\}$$
$$P'_i = \{<< x, y >, < x, z >>: yP_iz, x \in W\}$$

The notion of product can also be applied to classes of frames and logics.

We argue that it is reasonable to introduce bidimensional modal semantics to formalize the knowability operator: using a concrete case of the definition 1, we obtain:

Definition 2 *Given two frames*

$$F_1 =< W, R >$$
$$F_2 =< S, P >$$

where $F_1 =< W, R >$ is interpreted as an alethic frame such that W is a set of possible worlds and R is an accessibility relation between worlds, while $F_2 =< S, P >$ is interpreted as an epistemic frame composed by a set S of epistemic states and a plausibility relation P between epistemic states. The simple product of a modal and an epistemic frame *is defined as:*

$$F_1 \times F_2 =< W \times S, R', P' >$$

where:

$$< w, s > R' < w', s > \textit{ iff } wRw' \textit{ and } s \in S$$
$$< w, s > P' < w, s' > \textit{ iff } sPs' \textit{ and } w \in W$$

Clearly, just single-agent frames are considered here. The elements of $W \times S$ are called *modal-epistemic states* while R' and P' are called *two-dimensional accessibility relations for knowability.* The basic difference between alethic and epistemic frames is that the accessibility relation does not have the notion of agents indexed to it. In this sense it is impossible to use accessibility relations and plausibility relations in the same way . Using the intuition to model 2-dimensional possibility and knowledge, we can suppose that we have just one operator called knowability.

Given a product of a modal and an epistemic frames, add in the usual way a valuation v to the frame in order to obtain a model. Then the formal semantics for the two knowability operators $\boxminus_L$ and $\boxminus_G$ are defined as:

Definition 3 *(Local Knowability)*
$< w, s >\models \boxminus_L p$ *iff* $\exists w'(wRw' \wedge < w', s >\models p)$ *and* $\forall s'(sPs' \Rightarrow < w', s' >\models p)$.

Definition 4 *(Global Knowability)*
$< w, s >\models \boxminus_G p$ *iff* $\exists w'(wRw' \wedge < w', s >\models p$ *and* $(w' \models p)$ *and* $\forall s'(sPs'$
$\Rightarrow < w', s' >\models p)$ *and* $(s' \models p)$.

Adding to the two above clauses the following bidimensional classical valuations to connectives:
$< w, s >\models A$ iff $< w, s >\in v(A)$, for A atomic.
$< w, s >\models \neg p$ iff $< w, s >\not\models p$
$< w, s >\models p \wedge q$ iff $< w, s >\models p$ and $< w, s >\models q$
$< w, s >\models p \vee q$ iff $< w, s >\models p$ or $< w, s >\models q$
$< w, s >\models p \rightarrow q$ iff $< w, s >\not\models p$ or $< w, s >\models q$

These valuations are used to show that in each point $< w, s >$ it is possible to reason classically.

The knowability operator is introduced here with the aims to modeling the concept of “it is possible to know ”without using two modalities, but using instead just one complex modality. The global and local notions of knowability express the crutial property that agents may have different levels of knowledge, meaning that a proposition can be known in more than a single way.

It is important to note that the local knowability modality could be defined using 2-dimensional possibility and knowledge, but the same is not the case related to global knowability, which could be defined using both 2-dimensional modalities and 1-dimensional modalities. The basic difference between local and global knowability is that in the local case there are no interaction axioms between modalities of different dimensions, while in the global knowability it is indeed possible to have interaction between operators of different contexts. The next natural step is to find an axiomatization characterizing the two above semantical levels of knowability.

3 Axiomatic system for LK

A particular problem which arises in questions about combining modalities is that exposed in [4]: how to find an axiomatic system for a class of frames? An axiomatic system is proposed here in order to axiomatize the logic for the local knowability and global knowability. It is important to note that all these axioms could be represented in a powerset simple logic presentation, which is a better way to represent modal logics than simple logic system presentation, as showed in [2]. Note that the signature Σ_n of the LK is $\Sigma_1 = \{\boxminus_G, \boxminus_L, \neg\}$, for n=1.

Definition 5 *The* axiomatic system for the logic of knowability LK *is:*
Axioms
1. All tautologies from propositional classical logic;
2. $\boxminus_*(p \to q) \to (\boxminus_* p \to \boxminus_* q)$, $* \in \{L, G\}$
3. $\boxminus_G p \to \boxminus_L p$
Inference Rules:
4. MP
5. $\vdash p$ *then* $\vdash *p$, *for* $* \in \{\boxminus_L, \boxminus_G\}$

The axiomatic system above does not recognize in its language either $\diamond$ or K, but somehow conveys a common abstraction from such concepts. Both $\boxminus_L, \boxminus_G$ distributes over conjunction as the proof-system and the semantic show, given that it is possible to maintain a substantial part of classical reasoning for them.

4 Completeness result and transfer properties

A completeness proof is developed for the logic LK using the tools of [8] with some appropriate modifications in the notion of a perfect matrix.

Lemma 1 *Let* Γ *be a set of formulas.* Γ *is satisfiable iff there is a* $\boxminus$*-perfect matrix for* Γ*.*

Lemma 2 *Let* Γ *be a set of formulas.* Γ *is consistent iff there is a* $\boxminus$*-perfect matrix for* Γ*.*

In order to prove the two lemmas, we should make several adaptations in the methods and techniques proposed in [8], but saving the basic idea of the proof. In the following there is a general view on the completeness procedure: the first one establishes a relation between matrices and semantics. The second is a bridge linking matrices and axioms. To prove the first lemma, define a matrix and show that it is, in fact, a $\boxminus$-perfect matrix for the set Γ. Soundness is a consequence of the fact that the function in the matrix assigns to each pair $< w, s >$ a maximal consistent set. To prove the other direction we need to transform the $\boxminus$-perfect matrix into a model and then, by induction, to prove the truth lemma. To prove the second lemma, we must show that Γ is contained in a maximal consistent set. The other direction is the most difficult part. Given a consistent set, how is it possible to find its $\boxminus$-perfect matrix? The idea is to construct a matrix which should be a $\boxminus$-perfect matrix (without defects). Technical details of the proof will be given in the final version of the paper.

As a consequence of the two lemmas:

Theorem 1 *LK is sound and complete with respect to its class of modal-epistemic product models.*

In the scope of transferring theorems among logics it is already known that, if L_1 and L_2 are canonical logics, then the semantical product is also canonical. Given that every canonical logic is Kripke-complete, and given that the product logic is canonical, the desired result follows immediately. If the product is obtained from logics with known properties, then a general proof is also obtained by transferring properties of the given logics. This is not the case here, given that we do not know which are the logics used in the combination.

5 Conclusion

Methods for combining logics are an important tendency, as they are useful in the task of finding powerful logics able to map natural language (although the existence of the collapsing problem related to the most powerful mechanism for combining logics: fibring) . Such methods also have many applications in fields varying from philosophy to computer science. It is important to note that the process of combining, for example, two logical objects depends strongly on the nature of these objects. This means that we must start making a selection of a particular case of structure. In this sense, combining logics constitutes a chapter of something called Universal Logic, as defined in [1]. To illustrate these last three mysterious sentences, let me make reference to two articles: [3] and [2]. In [3], the authors show how to give categorical descriptions of methods, or mechanisms, for combining logics. But to realize this task, they must first choose a particular kind of structure: signatures, hilbertian calculi or interpretation systems for creating the categories SIG, HIL and INT in order to represent the mechanisms in categorical terms. The same happens in [2], but now in order to define synchronization, parameterization and fibring it is necessary first to choose a particular kind of structure: consequence systems, simple logic systems, powerset logic systems, etc. The present article shows a way in which it is possible to define a bidimensional modal logic for the knowability modality, without defining the kind of logic system associated, but, instead, departing from an approach by means of frames. The above construction is a clue that we should enter in the world of multi-dimensional modal logics and combination of logics to realize philosophical tasks.

In [5] some known logics are shown to be particular cases of fibring. Would it be possible to find two already known logics (one alethic and other epistemic, for example) showing that the logic LK can be obtained by some method of combination applied to these logics?

The problem of examining how to obtain the product of particular modal logics with epistemic logics is another task, as it is the question of understanding further properties related to our construction. Results concerning families of logics for knowability are still under investigation, but the task seems to be promising.

References

[1] J-Y. Béziau *Researchs on Universal Logic - excessivity, negation and sequents* PhD Thesis, Universite Denis Diderot, Paris 7.

[2] C. Caleiro, C. Sernadas, and A. Sernadas, *Mechanisms for combining logics,* Research report, Section of Computer Science, Department of Mathematics, Instituto Superior Tecnico, 1049-001 Lisboa, Portugal, 1999.

[3] C. Caleiro, C. Sernadas, and A. Sernadas, *Fibring of Logics as a Categorial Construction* J. Logic Comput. 9 (1999), no. 2, 149–179

[4] Gabbay, Dov M.; Shehtman, Valentin B, *Products of modal logics,* I. Log. J. IGPL 6 (1998), no. 1, 73–146.

[5] Gabbay, Dov M, *Fibring Logics* Oxford University Press, Clarendon Press, 1999.

[6] Kracht, Marcus; Wolter, Frank. *Properties of independently axiomatizable bimodal logics* J. Symbolic Logic 56 (1991), no. 4, 1469–1485

[7] Halpern, Joseph Y.; Moses, Yoram *A guide to completeness and complexity of modal logics of knowledge and belief* Artificial Intelligence 54 (1992), no. 3, 319–379.

[8] Marx, Maarten; Venema, Yde, *Multi-dimensional modal logic,* Applied Logic Series, 4. Kluwer Academic Publishers, Dordrecht, 1997. xiv+239 pp.

Institute of Logic
University of Neuchâtel
Neuchâtel - Switzerland
<alexandre.costa , alexandrecl >@ <unine.ch , cle.unicamp.br >
GTAL (UNICAMP, Brazil)